EJEMPLO 9:

Resolver el sistema $\begin{cases} x_1' = 4x_1 + 2x_2 + e^t \\ x_2' = 3x_1 + 3x_2 + e^t \end{cases}$.

RESOLUCIÓN:

La forma matricial es $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} . \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^t \\ e^t \end{pmatrix}$.

Los autovalores de $\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix}$ son 6 y 1 dado ambos son soluciones de:

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 6 = 0$$

El conjunto de autovectores es $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} ; \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\}$.

Luego, la diagonalización de la matriz expresa:

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix}$$

La solución matricial fundamental es:

$$e^{A_s t} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^{6t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix} = e^{\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} t} = \frac{1}{5} \begin{pmatrix} 2e^t + 3e^{6t} & -2e^t + 2e^{6t} \\ -3e^t + 3e^{6t} & 3e^t + 2e^{6t} \end{pmatrix}$$

La inversa de la matriz fundamental es:

$$e^{-A_s t} = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} e^{-6t} & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} \frac{3}{5} & \frac{2}{5} \\ -\frac{1}{5} & \frac{1}{5} \end{pmatrix} = e^{-\begin{pmatrix} 4 & 2 \\ 3 & 3 \end{pmatrix} t} = \frac{1}{5} \begin{pmatrix} 2e^{-t} + 3e^{-6t} & -2e^{-t} + 2e^{-6t} \\ -3e^{-t} + 3e^{-6t} & 3e^{-t} + 2e^{-6t} \end{pmatrix}$$

Usando estos datos, la solución complementaria es:

$$\vec{X}_c = e^{A_s t} \vec{K} = \frac{1}{5} \begin{pmatrix} 2e^t + 3e^{6t} & -2e^t + 2e^{6t} \\ -3e^t + 3e^{6t} & 3e^t + 2e^{6t} \end{pmatrix} . \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A(2e^t + 3e^{6t}) + B(-2e^t + 2e^{6t}) \\ A(-3e^t + 3e^{6t}) + B(3e^t + 2e^{6t}) \end{pmatrix}$$

$$\vec{X}_c = \begin{pmatrix} (2A - 2B)e^t + (3A + 2B)e^{6t} \\ (-3A + 2B)e^t + (3A + 2B)e^{6t} \end{pmatrix}$$

Solución propia:

$$\vec{X}_p = e^{A_s\,t}\int e^{-A_s\,t}\,\vec{f}(t)\,dt$$

$$\vec{X}_p = \frac{1}{5}\begin{pmatrix} 2e^{t}+3e^{6t} & -2e^{t}+2e^{6t} \\ -3e^{t}+3e^{6t} & 3e^{t}+2e^{6t} \end{pmatrix}\int \frac{1}{5}\begin{pmatrix} 2e^{-t}+3e^{-6t} & -2e^{-t}+2e^{-6t} \\ -3e^{-t}+3e^{-6t} & 3e^{-t}+2e^{-6t} \end{pmatrix}\begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix} dt$$

$$\vec{X}_p = \frac{1}{25}\begin{pmatrix} 2e^{t}+3e^{6t} & -2e^{t}+2e^{6t} \\ -3e^{t}+3e^{6t} & 3e^{t}+2e^{6t} \end{pmatrix}\int \begin{pmatrix} e^{-5t} \\ e^{-5t} \end{pmatrix} dt = \frac{1}{25}\begin{pmatrix} 2e^{t}+3e^{6t} & -2e^{t}+2e^{6t} \\ -3e^{t}+3e^{6t} & 3e^{t}+2e^{6t} \end{pmatrix}\begin{pmatrix} \frac{e^{-5t}}{-5} \\ \frac{e^{-5t}}{-5} \end{pmatrix}$$

$$\vec{X}_p = -\frac{1}{5}\begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$$

Luego:

$$\vec{X} = \vec{X}_c + \vec{X}_p = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} (2A-2B)e^{t} + (3A+2B)e^{6t} \\ (-3A+3B)e^{t} + (3A+2B)e^{6t} \end{pmatrix} - \frac{1}{5}\begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$$

Finalmente:

$$\begin{cases} X_1 = \dfrac{(2A-2B)}{5}e^{t} + \dfrac{(3A+2B)}{5}e^{6t} - \dfrac{1}{5}e^{t} \\ X_2 = \dfrac{(-3A+3B)}{5}e^{t} + \dfrac{(3A+2B)}{5}e^{6t} - \dfrac{1}{5}e^{t} \end{cases}$$

EJEMPLO 10:

Dado el sistema $\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2 \\ t \end{pmatrix}$, determinar su solución general.

RESOLUCIÓN:

Del ejemplo 7 resuelto en sistemas homogéneos se sabe que:

$$e^{\begin{pmatrix} 2 & -1 \\ 5 & -2 \end{pmatrix} t} = \begin{pmatrix} \frac{2+i}{5} & \frac{2-i}{5} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} \frac{-5i}{2} & \frac{1+2i}{2} \\ \frac{5i}{2} & \frac{1-2i}{2} \end{pmatrix} = \begin{pmatrix} cos(t) + 2sen\,(t) & -sen(t) \\ 5sen(t) & cos(t) - 2sen(t) \end{pmatrix}$$

Solución complementaria:

$$\vec{X}_c = e^{A_s\,t}\begin{pmatrix} A \\ B \end{pmatrix}$$

$$\vec{X}_c = \begin{pmatrix} cos(t) + 2sen\,(t) & -sen(t) \\ 5sen(t) & cos(t) - 2sen(t) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} A\,cos(t) + (2A - B)\,sen(t) \\ (5A - 2B)sen(t) + B\,cos(t) \end{pmatrix}$$

Solución propia:

$$\vec{X}_p = e^{A_s\,t}\int e^{-A_s\,t}\vec{f}(t)\,dt$$

$$\vec{X}_p = \begin{pmatrix} cos(t) + 2sen(t) & -sen(t) \\ 5sen(t) & cos(t) - 2sen(t) \end{pmatrix}\int\begin{pmatrix} cos(t) - 2sen(t) & sen(t) \\ -5sen(t) & cos(t) + 2sen\,(t) \end{pmatrix}\begin{pmatrix} -2 \\ t \end{pmatrix} dt$$

$$\vec{X}_p = \begin{pmatrix} cos(t) + 2\,sen(t) & -sen(t) \\ 5\,sen(t) & cos(t) - 2\,sen(t) \end{pmatrix}\int\begin{pmatrix} -cos(t) + 4\,sen\,(t) + t\,sen(t) \\ 10\,sen(t) + t\,cos(t) + 2t\,sen(t) \end{pmatrix} dt$$

$$\vec{X}_p = \begin{pmatrix} cos(t) + 2\,sen(t) & -sen(t) \\ 5\,sen(t) & cos(t) - 2sen(t) \end{pmatrix}\begin{pmatrix} -sen(\,t) - (4+t)cos(t) \\ (2+t)sen(t) - (9+2t)\,cos(\,t) \end{pmatrix} = \begin{pmatrix} -4 - t \\ -9 - 2t \end{pmatrix}$$

Solución general:

$$\vec{X} = \vec{X}_c + \vec{X}_p$$

$$\vec{X} = \vec{X}_c + \vec{X}_p = \begin{pmatrix} A\,cos(t) + (2A - B)\,sen(t) \\ (5A - 2B)\,sen(t) + B\,cos(t) \end{pmatrix} + \begin{pmatrix} -4 - t \\ -9 - 2t \end{pmatrix}$$

Finalmente:

$$\begin{cases} X_1 = A\,cos(t) + (2A - B)sen\,(t) - 4 - t \\ X_2 = (5A - 2B)\,sen(t) + B\,cos(t) - 9 - 2t \end{cases}$$